Matrices defining plane curves

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Abstract

For a given plane curve we review the explicit constructions of:

- the $1-1$ correspondence between linear determinantal representations and rank one (non-exceptional) bundles,
- the $1-1$ correspondence between pfaffian representations and rank two (non-exceptional with fixed determinant) bundles.

We try to generalize these results to construct determinantal representations which would encode rank 3 bundles as its cokernel.

Notation

- we work over the field $\mathbb{C}$, sometimes we restrict to $\mathbb{R}$,
- $F(x, y, z)$ homogeneous polynomial of degree d,
- $\mathcal{C}$ a smooth curve defined by $\{F(x, y, z) = 0\} \subset \mathbb{P}^2$.

Weierstrass cubic: $y^2 z = x^3 + p\,xz^2 + q\,z^3, \quad p, q \in \mathbb{C},$
or $y^2 z = x(x + \theta_1 z)(x + \theta_2 z), \quad \theta_1, \theta_2 \in \mathbb{C},$

Hesse cubic: $\lambda(x^3 + y^3 + z^3) = \mu\,xyz, \quad \lambda, \mu \in \mathbb{P}^1.$

Definition: Determinantal representation

It is very useful to represent F of degree d as a determinant of some matrix:
Find a $rd \times rd$ matrix with **linear** terms

$$M(x, y, z) = xA + yB + zC$$

such that

$$\det M(x, y, z) = c\,F^r(x, y, z), \quad \text{for some } c \neq 0.$$

Matrix M is a determinantal representation (of order r) of $\mathcal{C}$.
Clearly, multiplying a determinantal representation by invertible matrices preserves the underlying curve. Two determinantal representations M and M' are equivalent if

$$M' = XMY \quad \text{for some } X, Y \in \mathrm{GL}(rd, \mathbb{C}).$$

We consider determinantal representations up to equivalence.

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Pfaffian representation

Pfaffian representation is a representation of order 2 with all $2d \times 2d$ matrices being skew-symmetric. Study of pfaffian representations is strongly related to and motivated by determinantal representations. Every $d \times d$ determinantal representation A induces *decomposable pfaffian representation*

$$\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix}.$$

Note that the equivalence relation is well defined since

$$\begin{bmatrix} 0 & XMY \\ -(XMY)^t & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^t \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} X^t & 0 \\ 0 & Y \end{bmatrix}.$$

Theorem (Beauville, 2000: plane curves as determinants)

Let L be a line bundle of degree $g - 1 = \frac{1}{2}d(d-3)$ on $\mathcal{C}$ with $H^0(\mathcal{C}, L) = 0$. Then there exists a $d \times d$ linear matrix M such that $F = \det M$ and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^d \to L \to 0.$$

Conversely, let M be a $d \times d$ linear matrix such that $F = \det M$. Then the cokernel of $M : \mathcal{O}_{\mathbb{P}^2}(-2)^d \to \mathcal{O}_{\mathbb{P}^2}(-1)^d$ is a line bundle L on $\mathcal{C}$ of degree $g - 1$ with $H^0(\mathcal{C}, L) = 0$.

This is the famous $1-1$ correspondence between detreminantal representations of hypersurfaces and points on the Jacobian variety, first described in Cook and Thomas, Line bundles and homogeneous matrices, (1979).

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Theorem (Beauville, 2000: plane curves as pfaffians)

Let E be a rank 2 vector bundle on $\mathcal{C}$ with $\det E \cong \omega_{\mathcal{C}}$ and $H^0(\mathcal{C}, E) = 0$. Then E admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{2d} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1)^{2d} \to E \to 0,$$

where the matrix A is linear skew-symmetric and $F = \operatorname{Pf} A$.

Note that the condition $H^0(\mathcal{C}, E) = 0$ implies that E is semi-stable.

$\tilde{M} \cdot M = \det M \cdot \mathrm{Id}_d$

Theorem (Dolgachev's explicit $1-1$ correspondence)

Let $L \in \mathrm{Pic}(\mathcal{C})^{g-1} \setminus W_{g-1}$, where $\mathrm{Pic}(\mathcal{C})^{g-1}$ is the Picard variety of degree $g-1$ invertible sheaves (divisor classes) on $\mathcal{C}$ and W_{g-1} its subset of effective divisors. Then L and $L^{-1} \otimes \mathcal{O}_C(d-3)$ define a unique regular map

$$\mathcal{C} \to \left| H^0\left(\mathcal{C}, L(1)\right)^\vee \otimes H^0\left(\mathcal{C}, L(1)^{-1} \otimes \mathcal{O}_C(d-1)\right) \right|$$

which extends to a rational map on $\mathbb{P}^2$. In coordinates, this is the adjugate matrix of a determinantal representation of $\mathcal{C}$. Conversely, the kernel and cokernel (twisted by -1) of a given determinantal representation define L and L^{-1} as above.

We generalize Dolgachev's construction to rank 2:
Define the pfaffian adjoint of A to be the skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ & \ddots & (-1)^{i+j}\,\mathrm{Pf}^{ij} A & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

$\tilde{A} \cdot A = \operatorname{Pf} A \cdot \mathrm{Id}_{2d}$

Proposition (_ and Košir's explicit $1-1$ correspondence)

Let C be a smooth plane curve of degree d. To every rank 2 vector bundle $\mathcal{E}$ on C with properties

- (i) $h^0(C, \mathcal{E}) = 2d$,
- (ii) $H^0(C, \mathcal{E}(-1)) = 0$,
- (iii) $\det \mathcal{E} = \bigwedge^2 \mathcal{E} = \mathcal{O}_C(d-1)$

we can assign a pfaffian representation $A_{\mathcal{E}}$. In particular, isomorphic bundles induce equivalent representations. Conversly, the cokernel of a pfaffian representation of C is a rank 2 vector bundle on C with the above properties.

Buckley and Košir's explicit $1-1$ correspondence

Note that $\mathcal{E}(-1)$ has determinant $\mathcal{O}_C(d-3) = \omega_C$ and is exactly the rank 2 bundle from Beauville's pfaffian representation.

We will define a map ψ from C to the space of $2d \times 2d$ skew-symmetric matrices with entries from the space of homogeneous polynomials of degree $d-1$.
Let $U = H^0(C, \mathcal{E})$ be the $2d$ dimensional vector space of global sections of $\mathcal{E}$. Choose a basis $\{s_1, \ldots, s_{2d}\}$ for U and define

$$C \ni x \overset{\psi}{\mapsto} \sum_{1 \le i < j \le 2d} (s_i(x) \wedge s_j(x))(E_{ij} - E_{ji}) = \begin{bmatrix} 0 & & \\ & \ddots & s_i(x) \wedge s_j(x) \\ & \ddots & \\ & & 0 \end{bmatrix}.$$

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Since $s_i \wedge s_j \in \bigwedge^2 U$, by property (iii) the map ψ extends to

$$\Psi \colon \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\bigwedge^2 U)$$

given by a linear system of plane curves of degree $d-1$. In coordinates it equals to a $2d \times 2d$ skew-symmetric matrix with entries from the space of homogeneous polynomials of degree $d-1$. This is exactly the adjoint matrix of the pfaffian representation.

How can we generalize the identities of Cayley–Hamilton theorem: $\tilde{M} \cdot M = \det M \; \mathrm{Id}_d$ and $\tilde{A} \cdot A = \mathrm{Pf}\, A \; \mathrm{Id}_{2d}$?

$\tilde{A} \cdot A = \mathrm{Tri}\, A \; \mathrm{Id}_{3d}$

Let C be a smooth plane curve of degree d. To every rank 3 vector bundle $\mathcal{E}$ on C with the properties

(i) $h^0(C, \mathcal{E}(1)) = 3d$,

(ii) $H^0(C, \mathcal{E}) = 0$,

(iii) $\det \mathcal{E} = \bigwedge^3 \mathcal{E} = ??\mathcal{O}_C(d-3)$
$= ??$ *line bundle of degree* $\frac{3}{2}d(d-3)$,

we can assign a representation $A_{\mathcal{E}}$.

Background

Elliptic curves have profound influence in mathematics. Since ancient times they turn up in the most astonishing places, joining together algebra and geometry. Recently they have become popular in **number theory** (cryptography of elliptic curves), **optimization** (semidefinite programming SDP) and also in **theoretical physics** (mirror symmetry of elliptic curves).

The abundance of results is due to the following two classical facts for smooth plane cubics:

- It can be brought by a change of coordinates into the Weierstrass canonical form, or equivalently the Hesse canonical form.
- It can be equipped by a group law (induced by the Jacobian group variety).

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- It can be equipped by a group law (induced by the Jacobian group variety).

Atiyah: Vector bundles over an elliptic curve, 1957

Elliptic curves are of tame representation type according to Atiyah (1957). In particular, on a given cubic curve the number of indecomposable ACM bundles of rank r equals to the number of r-torsion points.

Theorem

For any $r > 0$ there exists an indecomposable vector bundle F_r defined inductively by $0 \to \mathcal{O}_C \to F_r \to F_{r-1} \to 0$, where $F_0 = \mathcal{O}_C,\ h^0(F_r) = 1$ and $F_r \cong F_r^\vee$.
Moreover, for any indecomposable bundle E of rank r and degree 0 there exists a line bundle L such that $E \cong F_r \otimes L$ and $\det E = L^{\otimes r}$.

Rank 2

Consider a rank 2 bundle E with $\det E = \mathcal{O}_C$ and $h^0(E) = 0$.

A decomposable E is of the form $L \oplus L^{-1}$ for $\mathcal{O}_C \neq L \in \mathrm{Pic}^0(C)$. This gives a block pfaffian representation.

An indecomposable bundle is isomorphic to the nontrivial extension of an (one of the three) even theta characteristics

$$0 \to \kappa_i \to E \to \kappa_i \to 0.$$

In other words, $\kappa_1, \kappa_2, \kappa_3$ are the $2-$torsion elements in $\mathrm{Pic}^0(C)$.

Ravindra and Tripathi, 2014 predicted indecomposable $3r \times 3r$ representations induced by repeated extensions of $r-$torsion line bundles.

Consider a minimal matrix A such that

$$\text{Coker}\left[\mathcal{O}_{\mathbb{P}^2}(-2)^{3r} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1)^{3r}\right]$$

is an indecomposable rank r bundle E with trivial determinant. Then $\det A = F^r$.

Furthermore, such E and A exist: for any rank r bundle E obtained by the Atiyah construction holds that E is $1-$regular, $h^0(E(-\mu)) = 0$ for $\mu \geq 0$ and $\det E = \mathcal{O}_C$. This implies that E has $3r$ minimal generators in degree 1 and a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{3r} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{3r} \to E \to 0.$$

Determinantal representations of Weierstrass cubics

We explicitly construct all determinantal representations of size $3 \times 3,\ 6 \times 6$ and 9×9.

Vinnikov explicitly parametrised determinantal representations by points on the affine curve:

Lemma

Consider the cubic in Weierstrass form:

$$F(x,y,z) = -yz^2 + x^3 + \alpha xy^2 + \beta y^3.$$

A complete set of determinantal representations of F is

$$x\,\mathrm{Id} + z \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} \frac{t}{2} & s & \alpha + \frac{3}{4}t^2 \\ 0 & -t & -s \\ -1 & 0 & \frac{t}{2} \end{pmatrix},$$

where $s^2 = t^3 + \alpha t + \beta$. Note that the last equation is exactly the affine part $F(t,1,s)$.

Main theorem

Let C be a smooth cubic in the Weierstrass form

$$F(x, y, z) = yz^2 - x(x - y)(x - \lambda y).$$

A complete set of pfaffian representations of F consists of three indecomposable representations and for the whole affine curve of decomposable representations:

$$x \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 1 & 0 & \frac{3t^2-2t(1+\lambda)-(1-\lambda)^2}{4} & 0 & \frac{t-1-\lambda}{2} \\ & 0 & 0 & 0 & -t & 0 \\ & & 0 & \frac{t-1-\lambda}{2} & 0 & -1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

for $t = 0, 1, \lambda$;

and

$$x\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 0 & \frac{3t^2-2t(1+\lambda)-(1-\lambda)^2}{4} & s & \frac{t-1-\lambda}{2} \\ & 0 & 0 & -s & -t & 0 \\ & & 0 & \frac{t-1-\lambda}{2} & 0 & -1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

where $s^2 = t(t-1)(t-\lambda)$. Note that the last equation is exactly the affine part $F(t, 1, s)$.

Theta characteristic

It is obvious that a 3×3 determinantal representation is **symmetric** if and only if $L \cong L^{-1}$. Such L is by definition a non-effective theta characteristic i.e., $L^{\otimes 2} \cong \omega_C$ and $H^0(C, L) = \{0\}$. Since every nonsingular cubic has exactly three even theta characteristics we get:

Corollary

A smooth cubic curve has three symmetric determinantal representations.

The following theorem constructs all three symmetric 3×3 representations:

Theorem (J. Harris, 1979, p. 696)

There exist precisely three points $(a, b) \in \mathbb{C}^2$ such that

$$aF = \mathrm{Hes}(bF + \mathrm{Hes}(F)),$$

where Hes *is the Hessian i.e., the determinant of the second partial derivatives matrix. The resulting three symmetric determinantal representations of F are inequivalent.*

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Using elementary transformations [Vinnikov] we can obtain all 3×3 determinantal representations of F from a given one.

How can we generalize the identities of Cayley–Hamilton theorem: $\tilde{M} \cdot M = \det M \ \mathrm{Id}_d$ and $\tilde{A} \cdot A = \mathrm{Pf}\, A \ \mathrm{Id}_{2d}$?

$\tilde{A} \cdot A = \mathrm{Tri}\, A \ \mathrm{Id}_{3d}$

Let C be a smooth plane curve of degree d. To every rank 3 vector bundle $\mathcal{E}$ on C with the properties

(i) $h^0(C, \mathcal{E}) = 3d$,

(ii) $H^0(C, \mathcal{E}(-\mu)) = 0$ *for* $\mu \geq 1$,

(iii) $\det \mathcal{E} = \bigwedge^3 \mathcal{E} = \mathcal{O}_C(d-1)$!!,

we can assign a representation $A_{\mathcal{E}}$.

Remark: for $\mathcal{E} = L_1 \oplus L_2 \oplus L_3$ the summands L_i do not induce a determinantal representations (unlike in the rank 2 case) since they are of wrong degree.

We can define a map ψ from C to the space of $3d \times 3d \times 3d$ matrices.
Choose a basis $\{s_1, \ldots, s_{3d}\}$ for $H^0(C, \mathcal{E})$ and define

$$x \overset{\psi}{\mapsto} \sum_{1 \leq i < j < k \leq 3d} (s_i(x) \wedge s_j(x) \wedge s_k(x))(E_{ijk} - E_{jik} + E_{jki} + E_{kij} - E_{kji} - E_{ikj})$$

which extends to

$$\Psi : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\bigwedge^3 U).$$

In coordinates it equals to a $3d \times 3d \times 3d$ matrix with entries from the space of homogeneous polynomials of degree $d - 1$.

- Rank of the constructed $3d \times 3d \times 3d$ matrix along C is 3.

For an analogue of the Cayley-Hamilton theorem for higher format matrices we need to define:

- determinant,
- multiplication,
- identity,
- trian.

Determinant

Let A be a $d \times d$ matrix. By definition,

$$\det(A) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} a_{i,\sigma(i)}.$$

A permutation

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & \cdots & d \\ j_1 & j_2 & j_3 & \cdots & j_d \end{bmatrix}$$

can be written as $\{(1, j_1), (2, j_2), \ldots, (d, j_d)\}$.

Then $\det(A) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod a_{1,j_1} a_{2,j_2} \cdots a_{d,j_d}.$

Pfaffian

Let A be a $2d \times 2d$ skew-symmetric matrix.

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} a_{\sigma(2i-1),\sigma(2i)},$$ where we sum over

$$\Pi = \{\sigma \in S_d \ : \ \sigma(2i-1) < \sigma(2i) \text{ and } \sigma(2i-1) < \sigma(2i+1).$$

A partition of $\{1, 2, \ldots, 2d\}$ into pairs can be written as $\{(i_1, j_1), (i_2, j_2), \ldots, (i_d, j_d)\}$ with $i_k < j_k$ and $i_k < i_{k+1}$.

Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2d \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_d \end{bmatrix}$ be the corresponding permutation.

$$\text{Then } \mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod a_{i_1,j_1} a_{i_2,j_2} \cdots a_{i_d,j_d}.$$

Trian

A partition of $\{1, 2, \ldots, 3d\}$ into triplets can be written as $\{(i_1, j_1, k_1), (i_2, j_2, k_2), \ldots, (i_d, j_d, k_d)\}$ with $i_m < j_m < k_m$ and $i_m < i_{m+1}$.

Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 3d \\ i_1 & j_1 & k_1 & i_2 & \cdots & k_d \end{bmatrix}$ be the corresponding permutation.

We could define

$$\mathrm{Tri}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod a_{i_1, j_1, k_1} a_{i_2, j_2, k_2} \cdots a_{i_d, j_d, k_d}.$$

In this case A is a matrix of format $3d \times 3d \times 3d$ representing a three dimensional tensor.

A. Ayyer: Determinants and Perfect Matchings, J. Combin. Theory (2013)

Ayyer gives a combinatorial interpretation of the determinant of a matrix as a generating function over Brauer diagrams. As a corollary he obtains Cayley's relation between determinants and Pfaffians.

Recall irreducible representations of S_n:

- the trivial 1-dim representation;
- the 1-dim sign representation $\epsilon : S_3 \to \pm 1$,
- S_3 also has the geometric (or standard) 2-dim representation.

The blocks of the trivial and the sign representation in $V^{\otimes n}$ are $\mathrm{Sym}^n V$ and $\wedge^n V$ respectively. In particular,

$$S_2 \text{ induces} \quad V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$$

and

$$S_3 \text{ induces} \quad V \otimes V \otimes V = \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \text{two copies of } V.$$

Corollary

A matrix $[v_{ijk}]$ of $t \times t \times t$ format can be written in a unique way as a sum of 6 matrices:

$$\begin{aligned}
v_{ijk} &= a_{ijk} + b_{ijk} + \omega c_{ijk} + 0 + e_{ijk} + 0,\\
v_{jki} &= a_{ijk} + b_{ijk} + c_{ijk} + 0 + \omega e_{ijk} + 0,\\
v_{kij} &= a_{ijk} + b_{ijk} + \omega^2 c_{ijk} + 0 + \omega^2 e_{ijk} + 0,\\
v_{jik} &= a_{ijk} - b_{ijk} + 0 + \omega d_{ijk} + 0 + f_{ijk},\\
v_{kji} &= a_{ijk} - b_{ijk} + 0 + d_{ijk} + 0 + \omega f_{ijk},\\
v_{ikj} &= a_{ijk} - b_{ijk} + 0 + \omega^2 d_{ijk} + 0 + \omega^2 f_{ijk}.
\end{aligned}$$

Here ω is the third root of unity.

Aim: Extend Ayer's construction to matrices with $c_{ijk} = d_{ijk} = e_{ijk} = f_{ijk} = 0$ to obtain $\det[b_{ijk}] = \mathrm{trian}^3$.

Cayley–Hamilton theorem generalizes to matrices of even format. They appear in Finsler geometry (relativity and gauge theory) and in fourth-rank gravity.

Tapia, 2008: Invariants and polynomial identities for higher rank matrices

Define $A^{-1} := \frac{1}{\det A} \frac{\partial \det A}{\partial A}$, which is in terms of components $A^{ijkl} = \frac{1}{\det A} \frac{\partial \det A}{\partial A_{ijkl}}$.

$$\text{Then,} \quad A^{i\,k_1 k_2 k_3} A_{j\,k_1 k_2 k_3} = \delta^i_j .$$

Note that a $3d \times 3d \times 3d$ matrix needs to be put into $3d \times 3d \times 3d \times 3d \times 3d \times 3d$ format to obtain a nonzero determinant.

Weierstrass canonical form

Theorem

By a projective change of coordinates, every irreducible curve can be brought into the Weierstrass form

$$y^2 z = x^3 + pxz^2 + qz^3, \quad p, q \in \mathbb{C}$$

or equivalently $y^2 z = x(x + \theta_1 z)(x + \theta_2 z), \quad \theta_1, \theta_2 \in \mathbb{C}.$

Moreover, every reduced curve is projectively equivalent to one of the

$$\begin{array}{ll} x^3,\ x^2y,\ xy(x+y),\ xyz & \text{or} \\ (\alpha x + \beta y + \gamma z)(x^2 - yz) & \text{for some } \alpha, \beta, \gamma \in \mathbb{C}. \end{array}$$

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Why do we want the Weierstrass canonical form?

Corollary

Any coordinate independent statement that holds for a Weierstrass cubic, holds for any irreducible cubic curve.

This implies:

- Determinantal representations of any cubic curve $\mathcal{C}$ are in one to one correspondence with affine points on $\mathcal{C}$.

Why do we want the Weierstrass canonical form?

Corollary

Any coordinate independent statement that holds for a Weierstrass cubic, holds for any irreducible cubic curve.

This implies:

- Determinantal representations of any cubic curve $\mathcal{C}$ are in one to one correspondence with affine points on $\mathcal{C}$.

Inflection point

Every irreducible cubic has inflection points:
$\{F = 0\} \cap \{\text{Hes}\, F = 0\} \subset \mathbb{P}^2$.

Proposition

If we find an inflection point on $\mathcal{C}$, we can put it into the Weierstrass form.

Change the coordinates so that the inflection point is $(0, 1, 0)$ and the inflection tangent is $z = 0$. Considering all possible monomials occurring in F yields the Weierstrass form.

Corollary

When the defining polynomial F is real, a real change of coordinates gives the Weierstrass form with $p, q \in \mathbb{R}$.

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Algorithm

- The enumerative problem of locating flexes of a plane cubic is solvable, since the corresponding Galois group is solvable [Harris, 1979].
- When $\mathcal{C}$ contains a rational point [Silverman and Tate, 1992] provided an algorithm that puts it into a Weierstrass form.

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